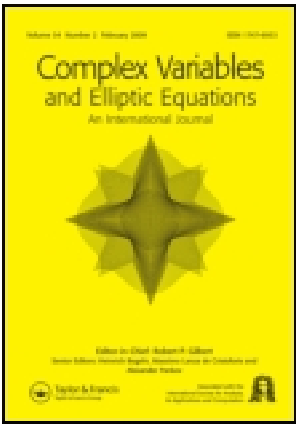

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q extension of Wielandt's Theorem

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q-extension of Wielandt's Theorem

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It is well known that the q-Gamma function cannot be characterised by its functional equation $\Gamma_q(z+1) = ((1-q^z)/(1-q))\Gamma_q(z)$ and the condition $\Gamma_q(1) = 1$. In 1980 Richard Askey showed in [1] that the additional assumption of logarithmic convexity yields the uniqueness of $\Gamma_q(x)$ for real $x > 0$. The goal of this note is to establish a q-extension of Wielandt's theorem which gives a characterization of $\Gamma_q(z)$ for all $z \in \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Keywords: q-Gamma function; Holomorphic functions

AMS Subject Classification: 33B15

1 THE FUNCTIONAL EQUATION

We consider a holomorphic function f in the half plane $A := \{z \in \mathbb{C}: \mathrm{Re}(z) > 0\}$ satisfying the equation:

$$f(z+1) = \frac{1-q^z}{1-q} f(z) \quad \text{for all} \quad z \in A \tag{1.1}$$

Throughout this note q belonging in $]0, 1[$.

By induction we obtain for all $n \in \mathbb{N}$ and all points $z \in A$

$$f(z+n+1) = (z)_q (z+1)_q (z+2)_q \cdots (z+n)_q f(z) \tag{1.2}$$

where $(z+n)_q = (1-q^{z+n})/(1-q)$

Now it is easily shown that: Every function f holomorphic in A and satisfying (1) admits a meromorphic extension $\hat{f}$ to $\mathbb{C}$. This function $\hat{f}$ is holomorphic in

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$\mathbb{C}-\{0,\,-1,\,-2,\ldots\}$; the point $-n$ is a pole of order ≤ 1 with residue

$$\frac{(-1)^n}{(n)_q!}q^{n(n+1)/2}\frac{q-1}{\log q}f(1) \tag{1.3}$$

where $(n)_q! = (n)_q(n-1)_q\cdots(1)_q$ and $(n)_q = (1-q^n)/(1-q)\ \forall n \in \mathbb{N}$.
In particular $\hat{f}$ is an entire function if and only if $f(1) = 0$.

Proof Take a point $z \in \mathbb{C}$ such that $-z \notin \mathbb{N}$. Then $\hat{z} := z+n+1 \in A$ for large $n \in \mathbb{N}$ and we may define the function $\hat{f}$ given by:

$$\hat{f}(z) := \frac{f(\hat{z})}{(z)_q(z+1)_q\cdots(z+n)_q}$$

Clearly this function is independant of the choice of n and we get a holomorphic function $\hat{f}$ in $\mathbb{C}-\{0,\,-1,\,-2,\ldots\}$ such that $\hat{f}_{/A} = f$. Furthermore

$$\lim_{z\longrightarrow -n}(z+n)\hat{f}(z) = \frac{(-1)^n}{(n)_q!}q^{n(n+1)/2}\frac{q-1}{\log q}f(1) \quad \text{for all} \quad n \in \mathbb{N}$$

This shows that $-n$ is a pole of $\hat{f}$ of order ≤ 1 with residue

$$\frac{(-1)^n}{(n)_q!}q^{n(n+1)/2}\frac{q-1}{\log q}f(1)$$

2 q-JACKSON'S FUNCTION

Our point of departure is the Jackson's function Γ_q given by:

$$\Gamma_q(z) = \frac{(q,q)_\infty}{(q^z,q)_\infty}(1-q)^{1-z} \tag{2.1}$$

with $0 < q < 1$ and $(z,q)_\infty = \prod_{k\geq 0}(1-zq^k)$
It is known that the q-Gamma function has the following properties:

- $\Gamma_q(z)$ is holomorphic in A
- $\Gamma_q(z)$ verified the functional equation given by

$$F(z+1) = (z)_qF(z) \quad \text{for all} \quad z \in A \quad \text{and} \quad F(1) = 1 \tag{2.2}$$

- Furthermore the inequations:

$$|(1-q)^{1-z}| = (1-q)^{1-\Re z}$$

and

$$|(1-q^{k+z})| \geq 1-q^{k+\Re z}$$

entails directly

$$|\Gamma_q(z)| \leq |\Gamma_q(\Re z)| \quad \text{for all} \quad z \in A$$

In particular $\Gamma_q(z)$ is bounded in every strip $\{z \in \mathbb{C}: a \leq \Re z \leq b\}$. with $0 < a < b < +\infty$.

3 THE MAIN RESULT

The main result of this note is given by the following theorem.

3.1 q-extension of Wielandt's Theorem

Let $F(z)$ be a holomorphic function in the right half plane A having the following two properties:

(a) $F(z+1) = (z)_q F(z)$ for all $z \in A$
(b) $F(z)$ is bounded in the strip $S_q := \{z \in \mathbb{C}: 2-q < \Re z < 1+q\}$ if $\frac{1}{2} < q < 1$ (respectively in the strip $S_q := \{z \in \mathbb{C}: 1+q < \Re z < 2-q\}$ if $0 < q < \frac{1}{2}$).

Then:

$$F(z) = \alpha\Gamma_q(z) \quad \text{in } A \text{ with} \quad \alpha = F(1)$$

Proof Without loss of generality, one can assume that $q \in]\frac{1}{2}, 1[$.

The function $f := F - \alpha\Gamma_q$ is holomorphic in A. From (a) we obtain:

$$f(z+1) = (z)_q f(z) \quad \text{for all} \quad z \in A$$

We have $f(1) = 0$, we conclude from (1.3) that f extends to an entire function $\hat{f}$. As the function Γ_q restricted to S_q is bounded by (2.3) and the function $f_{/S_q}$ is bounded by (b), thus f is bounded on S_q. This implies boundedness of $\hat{f}$ on $S_q^0 := \{z \in \mathbb{C}: 1-q \leq \Re z < q\}$. In fact, let $z \in S_q^0$. Then $z+1 \in S_q$; as f is bounded in S_q and $f(z+1) = (z)_q f(z)$. We deduce that: f is bounded in S_q^0.

We now consider the entire function denoted by s given by the following expression:

$$s(z) = q^{z(1-z)/2}\hat{f}(z)\hat{f}(1-z)$$

Since $\hat{f}(z)$ and $\hat{f}(1-z)$ take the same values on S_q^0 and that the expression $q^{z(1-z)/2}$ is bounded on S_q^0 . Then the function s is bounded on S_q^0.

On one hand let's consider now the strip $\Omega := \{z \in \mathbb{C}: \frac{1}{2} \leq \Re z \leq \frac{3}{2}\}$.. Then the function s verified the following properties given by the below assertion.

Assertion:

- s is an entire function
- $s(z+1) = -s(z) \quad \forall z \in \mathbb{C}$
- s has no essential singularity on $\partial\Omega$ the boundary of Ω.

We will only show the last point of the above assertion. In fact: If we suppose that the point a of $\partial\Omega$ is an essential singularity of s. Then by application of the Sohotsky's theorem [3, pp. 133–134], there is a sequence $z_n \to a$ such $s(z_n) \to \infty$. Absurd, since s is bounded on S_q^0 and S_q.

Remark 1 We can show the last point of the above assertion by application of the "big Picard theorem" [5, p. 332] which merely asserts that the image of every neighborhood of a is dense in the plane if the function has an essential singularity at a.

On the other hand, following [4, p. 678], as s is an entire function verified the above assertion, s has the following form:

$$s(z) = e^{i\pi z} P(e^{2i\pi z}) \quad \text{where } P(X) \in X^{-m}\mathbb{C}[X] \quad \text{for some} \quad m \in N \quad (*)$$

Then we assert that, by application of the Phragmen–Lindelof principle, s is bounded in $\Omega := \{z \in \mathbb{C}: \frac{1}{2} \leq \Re z \leq \frac{3}{2}\}$.

In fact: s is holomorphic in Ω and bounded on $\partial\Omega$ the boundary of Ω; since s is bounded in S_q^0 and S_q.

Put: $\varphi(z) = e^{z^2}$, we can easily verified that $\varphi(z)$ is holomorphic and bounded in Ω.

Furhtermore; from $(*)$ we show that:

$$\left|s(z)\right|\left|\varphi(z)\right|^{\nu} \to 0$$

as $z \to \infty$ in Ω; for all $\nu > 0$.

Indeed: let's suppose that the entire function s has the following form:

$$s(z) = e^{i\pi z} e^{-2mi\pi z}\left[a_0 e^{2ni\pi z} + a_1 e^{2(n-1)i\pi z} + \cdots + a_n\right]$$

where $a_i \in \mathbb{C}$ for $0 \leq i \leq n$ and $n \in \mathbb{N}$. Then, it's easy to show the following inequality:

$$|s(z)| \leq M e^{(2m-1)\pi y}\left[e^{-2n\pi y} + e^{-2(n-1)\pi y} + \cdots + 1\right]$$

where $z = x + iy$ and $M = \max\{a_i: 0 \leq i \leq n\}$. Subsequent we obtain:

$$\left|s(z)\right|\left|\varphi(z)\right|^{\nu} \leq M e^{(2m-1)\pi y}\left[e^{2n\pi y} + e^{2(n-1)\pi y} + \cdots + 1\right] e^{(9/4 - y^2)\nu}$$

for all $z = x + iy \in \Omega$ and $\nu > 0$.

As, we have:

$$M e^{(2m-1)\pi y}\left[e^{2n\pi y} + e^{2(n-1)\pi y} + \cdots + 1\right] e^{(9/4 - y^2)\nu} \to 0$$

when $z \to \infty$ in Ω; for all $\nu > 0$. Therefore the result follows. Thus, by application of the Phragmén–Lindelof principle, s is bounded in Ω.

It finally follows that the entire function s is bounded in $\mathbb{C}$ and hence constant by Liouville's theorem.

Then we get:

$$s(z) = s(1)$$

This implies

$$\hat{f}(z) = 0$$

Therefore, we conclude that:

$$F(z) = \alpha\Gamma_q(z) \quad \text{in } A \text{ with} \quad \alpha = F(1).$$

Remark 2 When q tend 1^- we obtain the classical Wielandt's theorem [2].

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